

Néron's pairing and relative algebraic equivalence

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Abstract

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K . Let X_K be a projective smooth and geometrically connected scheme over K . Néron defined a canonical pairing on X_K between 0-cycles of degree zero and divisors which are algebraically equivalent to zero. When X_K is an abelian variety, and if one restricts to those 0-cycles supported by K -rational points, Néron gave an expression of his pairing involving intersection multiplicities on the Néron model A of A_K over R . When X_K is a curve, Gross and Hriljac gave independantly an analogous description of Néron's pairing, but for arbitrary 0-cycles of degree zero, by means of intersection theory on a proper flat regular R -model X of X_K .

In this article, we show that these intersection computations are valid for an arbitrary scheme X_K as above and arbitrary 0-cycles of degree zero, by using a proper flat *normal and semi-factorial* model X of X_K over R . When $X_K = A_K$ is an abelian variety, and $X = \bar{A}$ is a semi-factorial compactification of its Néron model A , these computations can be used to study the algebraic equivalence on \bar{A} . We then obtain an interpretation of Grothendieck's duality for the Néron model A , in terms of the Picard functor of \bar{A} over R .

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1 Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K . Let X_K be a projective smooth and geometrically connected scheme over K . Denote by $Z_0^0(X_K)$ the group of 0-cycles of degree zero on X_K , and by $\text{Div}^0(X_K)$ the group of divisors which are algebraically equivalent to zero on X_K . For each $c_K \in Z_0^0(X_K)$ and $D_K \in \text{Div}^0(X_K)$ with disjoint supports, Néron attached canonically a rational number

$$\langle c_K, D_K \rangle \in \mathbb{Q},$$

by using the unique (up to constant) Néron function associated to D_K . This defines a bilinear pairing $\langle \cdot, \cdot \rangle$: see [N].

Suppose first that $X_K = A_K$ is an abelian variety, and denote by A its Néron model over R . By definition of A , any K -rational point of A_K extends to a section of A over R . Then, if c_K is supported by K -rational points, Néron showed that the pairing attached to A_K can be decomposed as follows:

$$\langle c_K, D_K \rangle = i(c_K, D_K) + j(c_K, D_K),$$

where $i(c_K, D_K)$ is the intersection multiplicity $(\overline{c_K}, \overline{D_K}) \in \mathbb{Z}$ of the schematic closures in A , and $j(c_K, D_K) \in \mathbb{Q}$ only depends on the specialization of c_K on the group Φ_A of connected components of the special fiber A_k : see [N] III 4.1 and [La] 11.5.1.

Suppose now that X_K is a curve, and denote by X a proper flat regular model of X_K over R . Let M be the intersection matrix of the special fiber X_k of X/R : if $\Gamma_1, \dots, \Gamma_\nu$ are the irreducible components of X_k equipped with their reduced scheme structure, the $(i, j)^{\text{th}}$ entry of M is the intersection number $(\Gamma_i \cdot \Gamma_j)$. Let $D_K \in \text{Div}^0(X_K)$ and let $\overline{D_K}$ be its closure in X . Computing the degree $(\overline{D_K}, \Gamma_i)$ of $\overline{D_K}$ along each Γ_i , we get a vector $\rho(\overline{D_K}) \in \mathbb{Z}^\nu$. Next, as a consequence of intersection theory on X , there exists a vector $V \in \mathbb{Q}^\nu$ such that $\rho(\overline{D_K}) = MV$. Still denote by V the \mathbb{Q} -linear combination of the Γ_i where the coefficient of Γ_i is the i^{th} entry of V . Then, for any $c_K \in Z_0^0(X_K)$ whose support is disjoint from the one of D_K , the following formula holds:

$$\langle c_K, D_K \rangle = (\overline{c_K}, \overline{D_K}) + (\overline{c_K}, (-V)),$$

where the second intersection number is defined by \mathbb{Q} -linearity from the $(\overline{c_K}, \Gamma_i)$. See [G], [H] and [La2] III 5.2. Now let J_K be the Jacobian of X_K and let J be its Néron model over R . Following the point of view of Bosch and Lorenzini ([BL] 4.3), it results from Raynaud's theory of the Picard functor $\text{Pic}_{X/R}$ ([R] 8) that the term $(\overline{c_K}, (-V))$ only depends on the specialization of $(c_K) \in J_K(K)$ into the group of components Φ_J of J_k .

In section 2, we provide a unified approach to these both descriptions of Néron's pairing. More precisely, for an arbitrary proper geometrically normal and geometrically connected scheme X_K , there always exists some proper flat normal semi-factorial model X of X_K over R : see [P]. Recall that X/R is *semi-factorial* if the restriction homomorphism on Picard groups $\text{Pic}(X) \rightarrow \text{Pic}(X_K)$ is surjective. Note that a regular model is semi-factorial. Using the theory of the Picard functor of semi-factorial models, we define a pairing $[\cdot, \cdot]$ on X_K involving intersection multiplicities on X (Definition 2.1). It turns out that this pairing only depends on X_K , and in fact coincides with Néron's pairing when X_K is projective smooth (Theorem 2.3). If $X_K = A_K$ is an abelian variety and $X = \overline{A}$ is a semi-factorial compactification of its Néron model A , then we recover the above description on the regular open subset $A \subseteq \overline{A}$. If X_K is a curve and X a proper flat regular model of X_K , then the intersection matrix of X_k is defined, and we exactly get Gross-Hriljac's formula.

In section 3, we consider an abelian variety A_K , with dual A'_K . By definition, the abelian variety A'_K parametrizes divisors on A_K which are algebraically equivalent to zero, that is $A'_K = \text{Pic}_{A_K/K}^0$. Then the Barsotti-Weil Theorem asserts that the forgetful map $\text{Ext}^1(A_K, \mathbb{G}_{m,K}) \rightarrow A'_K$ is an isomorphism. Starting from this, Grothendieck conjectured the behavior of the duality at the level of Néron models as follows. Let A'/R be the Néron model of A'_K . The Poincaré biextension of $A_K \times_K A'_K$ by $\mathbb{G}_{m,K}$ induces a canonical pairing between the component groups Φ_A and $\Phi_{A'}$ of the special fibers A_k and A'_k , with values in \mathbb{Q}/\mathbb{Z} . The conjecture is that this pairing is *perfect*: see [SGA 7] IX 1.3. Equivalently ([B] 5.1), the Barsotti-Weil isomorphism extends over R to an *isomorphism* $\text{Ext}^1(A, \mathbb{G}_{m,R}) \rightarrow (A')^0$, where

the 0 stands for the identity component. This statement remains open in equal characteristic $p > 0$; see however the introduction of [BL] for a detailed list of the known cases, and also [Loe].

Here, we give an equivalent form of this duality statement, in terms of *algebraic equivalence on \overline{A}* (Theorem 3.4). It has to be seen as the extension over R of the defining duality isomorphism $A'_K = \text{Pic}_{A_K/K}^0$. To achieve this reformulation of Grothendieck's conjecture, two ingredients are needed. Firstly, the link between the duality pairing and Néron's pairing, established by Bosch and Lorenzini ([BL] 4.4). Secondly, intersection theory on \overline{A} for Néron's pairing on A_K . Here we crucially make use of this pairing for 0-cycles of degree zero supported by *nonrational* points (Proposition 3.6).

In section 4, we improve slightly the computations of Néron's and Grothendieck's pairing for Jacobians worked out by Bosch and Lorenzini in [BL] 4.6, and by Lorenzini in [Lor] 3.4 (Proposition 4.1).

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2 Néron's pairing and intersection multiplicities

Let R be a discrete valuation ring with fraction field K and residue field k . **We assume R complete and k algebraically closed.**

For the definition and properties of Néron's pairing, we will refer to the Lang's book [La], especially to the Theorem 3.5 of chapter 11. A Néron's pairing over a complete field L will always be computed using the *normalized* discrete valuation on L , that is *with value group \mathbb{Z}* .

Moreover, let us adopt the following terminology: a *divisor* on a scheme will always be a *Cartier divisor*.

2.1 A canonical pairing computed on semi-factorial models

Let X_K be a proper geometrically normal and geometrically connected scheme over K . By [P] 2.6, there exists a model X/R of X_K , that is an R -scheme with generic fiber X_K , which is proper flat normal and *semi-factorial*: every invertible sheaf on X_K can be extended to an invertible sheaf on X . To each 0-cycle $c_K \in Z_0^0(X_K)$ and divisor $D_K \in \text{Div}^0(X_K)$ with support disjoint from the one of c_K , we will attach a number $[c_K, D_K]_X \in \mathbb{Q}$ using intersection multiplicities on X . For this purpose, let us first recall some definitions and one result.

Intersection multiplicities. Let X/R be a proper flat scheme over R . Let c_K be a 0-cycle on the generic fiber X_K , and denote by $\overline{c_K}$ its schematic closure in X . On the other hand, let Δ be a divisor on X whose support does not meet that of c_K . The *intersection multiplicity* $(\overline{c_K}, \Delta)$ of $\overline{c_K}$ and Δ on X is defined as follows. Let x_K be a point of the support of c_K . Let Z be its schematic closure in X . This is an integral scheme, finite and flat over R , which is local because R is henselian. Set $x_k := Z \cap X_k$. If $f \in K(X)$ is a local equation for Δ in the neighborhood of x_k , then $(\overline{c_K}, \Delta)_{x_k}$ is the order of $f|_Z$ at x_k : writing $f|_Z = a/b$ with regular $a, b \in \mathcal{O}(Z)$, then

$$(\overline{c_K}, \Delta)_{x_k} = \text{length}_{\mathcal{O}(Z)}(\mathcal{O}(Z)/(a)) - \text{length}_{\mathcal{O}(Z)}(\mathcal{O}(Z)/(b))$$

([F] page 8). The whole intersection multiplicity $(\overline{c_K}, \Delta)$ is defined by \mathbb{Z} -linearity.

Let us also give another description of $(\overline{c_K}.\Delta)_{x_k}$, which will be useful in the sequel. As R is excellent, the normalization $\tilde{Z} \rightarrow Z$ is finite. Moreover, as k is algebraically closed,

$$\text{length}_{\mathcal{O}(Z)}(\mathcal{O}(Z)/(a)) = \text{length}_R(\mathcal{O}(Z)/(a)),$$

for any regular $a \in \mathcal{O}(Z)$, and the same formula holds with Z replaced by \tilde{Z} (*loc. cit.* Appendix A.1.3). But

$$\text{length}_R(\mathcal{O}(Z)/(a)) = \text{length}_R(\mathcal{O}(\tilde{Z})/(a))$$

for any regular $a \in \mathcal{O}(Z)$ (see [BLR], end of page 237). Thus, if $f \in K(X)$ is a local equation for Δ in the neighborhood of x_k , we have obtained that

$$(\overline{c_K}.\Delta)_{x_k} = \begin{cases} \text{length}_{\mathcal{O}(\tilde{Z})}(\mathcal{O}(\tilde{Z})/(f)) & \text{if } f|_{\tilde{Z}} \in \mathcal{O}(\tilde{Z}), \\ -\text{length}_{\mathcal{O}(\tilde{Z})}(\mathcal{O}(\tilde{Z})/(f^{-1})) & \text{otherwise.} \end{cases}$$

Algebraic equivalence and τ -equivalence. ([R] 3.2 d) and [SGA 6] XIII 4) If G is a commutative group scheme locally of finite type over a field, the *identity component* G^0 of G is the open subscheme of G whose underlying topological space is the connected component of the identity element of G . The τ -*component* of G is open subgroup scheme G^τ of G which is the inverse image of the torsion subgroup of G/G^0 . When G is a commutative group foncteur over a scheme T , whose fibers are representable by schemes locally of finite type, the *identity component* (resp. τ -*component*) of G is the subfunctor G^τ of G whose fibers are the G_t^0 , $t \in T$ (resp. G_t^τ , $t \in T$). Note that $G^0 \subseteq G^\tau$.

Let $Z \rightarrow T$ be a proper morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_Z -module. The sheaf \mathcal{L} is said to be *algebraically equivalent to zero* (resp. τ -*equivalent to zero*) if its image into $\text{Pic}_{Z/T}(T)$ belongs to the subgroup $\text{Pic}_{Z/T}^0(T)$ (resp. $\text{Pic}_{Z/T}^\tau(T)$), that is $\mathcal{L}_t \in \text{Pic}_{X_t/t}^0(t)$ (resp. $\mathcal{L}_t \in \text{Pic}_{X_t/t}^\tau(t)$) for all $t \in T$. If D is a divisor on Z , it is *algebraically equivalent to zero* (resp. τ -*equivalent to zero*) if the associated invertible sheaf $\mathcal{O}_Z(D)$ is. Denoting by $\text{Div}^\tau(Z)$ the group of divisors on Z which are τ -equivalent to zero, we get $\text{Div}^0(Z) \subseteq \text{Div}^\tau(Z)$.

Algebraic equivalence and semi-factoriality. ([P] 3.11) Let X/R be a proper flat *semi-factorial* R -scheme. Suppose that the generic fiber X_K is geometrically normal and geometrically connected. Let A/S be the Néron model of the Picard variety $\text{Pic}_{X_K/K, \text{red}}^0$ of X_K and let n be the exponent of the component group of the special fiber of A/S . Then, for any divisor D_K on X_K which is algebraically equivalent to zero, there exists a divisor Δ on X which is algebraically equivalent to zero and whose generic fiber Δ_K is equal to nD_K .

Definition 2.1. Let X_K be a proper geometrically normal and geometrically connected scheme over K . Let X/R be a proper flat normal and semi-factorial model of X_K over R .

Consider $c_K \in Z_0^0(X_K)$ and $D_K \in \text{Div}^\tau(X_K)$ with disjoint supports. Choose $(n, \Delta) \in (\mathbb{Z} \setminus \{0\}) \times \text{Div}^\tau(X)$ such that $\Delta_K = nD_K$. Denoting by $\overline{c_K}$ the schematic closure of c_K in X , set

$$[c_K, D_K]_X = \frac{1}{n}(\overline{c_K}.\Delta) \in \mathbb{Q}.$$

This definition makes sense because the rational number $(1/n)(\overline{c_K}.\Delta)$ does not depend on the choice of (n, Δ) . Indeed, if (n', Δ') is another choice, the divisor $n'\Delta - n\Delta'$ is τ -equivalent to zero on X and equal to zero on X_K . Thus, as X is

normal, this difference is a rational multiple of the principal divisor X_k ([R] 6.4.1 3)). Now note that $(\overline{c_K} \cdot X_k)$ is equal to the degree of c_K , which is zero.

Next, one checks easily that the symbol $[\ , \]_X$ is *bilinear* (in its definition domain). To prove that this pairing does not depend on the choice of X , we will use the following lemma, which is an immediate consequence of the projection formula in intersection theory ([F] 2.3 (c)).

Lemma 2.2. *Let $\varphi : X \rightarrow X'$ be a morphism of proper S -schemes, which are flat, normal and semi-factorial, with geometrically normal and geometrically connected generic fibers. Let $c_K \in Z_0^0(X_K)$, and $D'_K \in \text{Div}^\tau(X'_K)$ whose support does not meet the one of $(\varphi_K)_*c_K$. The following equality holds:*

$$[c_K, (\varphi_K)^*D'_K]_X = [(\varphi_K)_*c_K, D'_K]_{X'}.$$

So, in the situation of Definition 2.1, let X' be another proper flat normal semi-factorial R -model of X_K . Consider the graph Γ of the rational map $X \dashrightarrow X'$ induced by the identity on the generic fibers. This is a closed subscheme of $X \times_S X'$, proper and flat over R , with generic fiber isomorphic to X_K . Applying [P] 2.6, we can find an R -scheme \tilde{X} which is proper flat normal and semi-factorial, together with an R -morphism $\tilde{X} \rightarrow \Gamma$ which is an isomorphism on generic fibers. Composing with the two projections from $X \times_S X'$ to X and X' , we get arrows

$$\begin{array}{ccc} & \tilde{X} & \\ \swarrow & & \searrow \\ X & & X' \end{array}$$

which are isomorphisms on generic fibers. Now, the above lemma shows that the pairings $[\ , \]_X$ and $[\ , \]_{X'}$ both coincide with $[\ , \]_{\tilde{X}}$. In conclusion, the pairing $[\ , \]_X$ only depends on X_K , and not on the choice of X . In the sequel, it will be denoted by $[\ , \]$.

2.2 Comparison with Néron's pairing

Let v the *normalized* valuation on K , which maps any uniformizing element of R to $1 \in \mathbb{Z}$. We fix an algebraic closure \overline{K} of K , and we still denote by v the unique valuation on \overline{K} extending v . Néron's pairing will always be computed with respect to v .

Let us state the common generalization of [N] III 4.1, Gross [G], Hriljac [H], [La2] III 5.2 and Bosch-Lorenzini [BL] 4.3, over a complete discrete valuation ring R with algebraically closed residue field k and fraction field K . Note that the group $(\mathbb{R}, +)$ being divisible, Néron's pairing is naturally defined for divisors which are only τ -equivalent to zero. So its definition domain is the same as the one of $[\ , \]$.

Theorem 2.3. *For every projective smooth and geometrically connected scheme over K , the pairing $[\ , \]$ defined in subsection 2.1 coincide with Néron's pairing $\langle \ , \ \rangle$. In particular, the pairing $[\ , \]$ extends Néron's pairing to every proper geometrically normal and geometrically connected scheme over K .*

Before proving the theorem, let us note the following consequence.

Corollary 2.4. *Let X_K be a proper geometrically normal and geometrically connected scheme over K . Let n be the exponent of the component group of the special fiber of the Néron model of the Picard variety $\text{Pic}_{X_K/K, \text{red}}^0$. Then Néron's pairing attached to X_K takes values in $\frac{1}{n} \mathbb{Z}$.*

This provides a refinement of [N] III 4.2, where the integer n is replaced by some multiple which is nontrivial if $X_K(K)$ is empty or if the abelian variety $\text{Pic}_{X_K/K, \text{red}}^0$ is not principally polarized. In [MT] (1.5) and (2.3), or [La] 11.5.1, this is already proved in the case where X_K is an abelian variety and the 0-cycles are supported by rational points. In this context, this is also a consequence of [BL] 4.4. On the other hand, the latter shows that Néron's pairing can take the value $1/n$, for example when X_K is an elliptic curve (see [BL] Example 5.8).

Let us go back to Theorem 2.3. To prove that both pairings coincide, it is enough to check it for divisors which are algebraically equivalent to zero. Then, to make the desired comparison, we will use the characterisation of Néron's pairing given in [La] 11.3.2 and recalled below.

An element c_K of $Z_0^0(X_K)$ can be written uniquely as a difference of two positive 0-cycles with disjoint supports: $c_K = c_K^+ - c_K^-$. Denoting by \deg the degree of a 0-cycle, let us set

$$\deg^+ c_K := \deg(c_K^+) = \deg(c_K^-) \geq 0.$$

This integer is called the *positive degree* of c_K .

Lemma 2.5. ([La] 11.3.2) *Suppose that for each projective smooth and geometrically connected scheme X_K over K , we are given a bilinear pairing*

$$\begin{aligned} Z_0^0(X_K) \times \text{Div}^0(X_K) &\rightarrow \mathbb{R} \\ (c_K, D_K) &\mapsto \delta(c_K, D_K) \end{aligned}$$

such that the following properties are true:

1. *If D_K is a principal divisor on X_K , then $\delta(c_K, D_K) = 0$.*
2. *Let $\varphi_K : X_K \rightarrow X'_K$ be a K -morphism. For all $c_K \in Z_0^0(X_K)$, and for all $D'_K \in \text{Div}^0(X'_K)$ whose support does not meet that of the 0-cycle $(\varphi_K)_* c_K$, the following equality holds*

$$\delta(c_K, (\varphi_K)^* D'_K) = \delta((\varphi_K)_* c_K, D'_K).$$

3. *Fix $D_K \in \text{Div}^0(X_K)$. As c_K varies in $Z_0^0(X_K)$, with bounded positive degree, the values $\delta(c_K, D_K)$ are bounded.*

Then $\delta(c_K, D_K) = 0$ for all c_K, D_K and X_K .

Proof of Theorem 2.3. Starting from the existence of *Néron functions* (e.g. see [La] Chapter 11), let us recall the definition of Néron's pairing. Let

$$c_K = \sum_i n_i [x_{K,i}] \in Z_0^0(X_K)$$

and $D_K \in \text{Div}^0(X_K)$ whose support $\text{Supp}(D_K)$ does not contain any of the $x_{K,i}$. Let $\lambda_{D_K} : (X_K - \text{Supp}(D_K))(\overline{K}) \rightarrow \mathbb{R}$ be a Néron function associated to D_K . For each i , the scheme $x_{K,i} \otimes_K \overline{K}$ is supported by some \overline{K} -points $x_{\overline{K},j_i}$, $j_i = 1, \dots, s_i$, where s_i is the separable degree of $K(x_{K,i})/K$. Denoting by l_i the inseparable degree of $K(x_{K,i})/K$, then

$$\lambda_{D_K}(x_{K,i}) := \sum_{j_i=1}^{s_i} l_i \lambda_{D_K}(x_{\overline{K},j_i}) \quad \text{and} \quad \langle c_K, D_K \rangle := \sum_i n_i \lambda_{D_K}(x_{K,i}).$$

The real number $\langle c_K, D_K \rangle$ is well-defined because λ_{D_K} is unique up to constant and c_K has degree zero. It follows from [N] III 4.2 that this number is *rational*.

However, we will not need this fact, and this will be a consequence of the theorem (see Corollary 2.4 above).

Comparison of the pairings for a principal divisor D_K .

Let us keep the previous notation, and suppose that $D_K = \text{div}_{X_K} f$ for a nonzero $f \in K(X_K)$. Let $z \in (X_K - \text{Supp}(\text{div}_{X_K} f))(\overline{K})$, mapping to a closed point $x_K \in X_K$. The evaluation of f at z is defined by the pull-back $z^* : \mathcal{O}_{X_K, x_K} \rightarrow \overline{K}$, that is, $f(z) := z^* f$. The formula $\lambda_f(z) = v(f(z))$ then defines a Néron function for the divisor $\text{div}_{X_K} f$.

Fix an i . There is a 1-1 correspondance between the $x_{\overline{K}, j_i}$ and the K -embeddings of the residue field extension $K(x_{K,i})/K$ into \overline{K}/K . By pulling-back the valuation v , each of these embeddings induces a valuation on $K(x_{K,i})$. However, as R is complete, these valuations are equal to the unique valuation on $K(x_{K,i})$ which extends the normalized valuation on K , and that we can also denote by v . Consequently,

$$\lambda_f(x_{K,i}) = \sum_{j_i=1}^{s_i} l_i v(f(x_{K,i})) = [K(x_{K,i}) : K] v(f(x_{K,i}))$$

where $f(x_{K,i})$ is the image of f by the canonical surjection $\mathcal{O}_{X_K, x_{K,i}} \rightarrow K(x_{K,i})$.

Now, take the schematic closure Z_i of $x_{K,i}$ in X , denote by \widetilde{Z}_i its normalization and set $x_{k,i} = X_k \cap Z_i$. The ring $\mathcal{O}(\widetilde{Z}_i)$ is a discrete valuation ring with fraction field $K(x_{K,i})$. So it is precisely the valuation ring of v in $K(x_{K,i})$. As k is algebraically closed, its ramification index over R is equal to $[K(x_{K,i}) : K]$. From this observation, we get

$$v(f(x_{K,i})) = \begin{cases} 1/[K(x_{K,i}) : K] \text{length}_{\mathcal{O}(\widetilde{Z}_i)}(\mathcal{O}(\widetilde{Z}_i)/(f)) & \text{if } f|_{\widetilde{Z}_i} \in \mathcal{O}(\widetilde{Z}_i), \\ -1/[K(x_{K,i}) : K] \text{length}_{\mathcal{O}(\widetilde{Z}_i)}(\mathcal{O}(\widetilde{Z}_i)/(f^{-1})) & \text{otherwise.} \end{cases}$$

We have thus obtained $[K(x_{K,i}) : K] v(f(x_{K,i})) = (\overline{c_K} \cdot \text{div}_X f)_{x_{k,i}}$ (recall the beginning of subsection 2.1). But $\text{div}_X f$ is a divisor on X which is τ -equivalent to zero and extends $\text{div}_{X_K} f$. The desired equality $\langle c_K, \text{div}_{X_K} f \rangle = [c_K, \text{div}_{X_K} f]$ follows.

The pairing $\delta(\cdot, \cdot)$.

Both $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ are bilinear in their definition domain, and they coincide for principal divisors. Using a moving lemma on the projective smooth scheme X_K (e.g. [Li] 9.1.11), we see that

$$\delta(c_K, D_K) := \langle c_K, D_K \rangle - [c_K, D_K]$$

is well-defined on the *whole* product $Z_0^0(X_K) \times \text{Div}^0(X_K)$. Condition 1 of Lemma 2.5 is satisfied by δ .

Condition 2 of 2.5 is satisfied by $\delta(\cdot, \cdot)$.

We know that Néron's pairing enjoys the functoriality described in 2.5 2. So we have to see that the pairing $[\cdot, \cdot]$ has the same property. Let X/R (resp. X'/R) be a proper flat normal semi-factorial model of X_K (resp. X'_K). Consider the graph Γ of the rational map $X \dashrightarrow X'$ defined by φ_K . Applying [P] 2.6 to Γ , we obtain a proper flat normal semi-factorial \widetilde{X}/R and R -morphisms

$$\begin{array}{ccc} & \widetilde{X} & \\ \alpha \swarrow & & \searrow \beta \\ X & & X' \end{array}$$

such that on the generic fibers, α is an isomorphism and β coincide with φ_K . In particular, the pairing $[\cdot, \cdot]$ for X_K can be computed on \widetilde{X} , and the desired functoriality follows from Lemma 2.2 applied to β .

Condition 3 of 2.5 is satisfied by $\delta(\ , \)$.

Denote by \overline{R} the valuation ring of v in \overline{K} .

Fix $D_K \in \text{Div}^0(X_K)$. Let $(n, \Delta) \in (\mathbb{Z} \setminus \{0\}) \times \text{Div}^\tau(X_K)$ satisfying $\Delta_K = nD_K$. Represent the divisor Δ by a family $(U_t, g_t)_{t=1, \dots, m}$, where the U_t are affine open subsets of X and the g_t are rational functions on X . Let E_t be the set of \overline{K} -points of X_K which extend to \overline{R} -points of U_t . As X is proper over R , we see that $X(\overline{K}) = \cup_{t=1}^m E_t$. The family $(U_{t,K}, g_t)_{t=1, \dots, m}$ represents the divisor nD_K on X_K . Let us choose a Néron function λ_{nD_K} on X_K . By definition, we can find some v -continuous locally bounded functions $\alpha_t : U_{t,K}(\overline{K}) \rightarrow \mathbb{R}$ such that

$$\lambda_{nD_K}(z) = v(g_t(z)) + \alpha_t(z)$$

for all $z \in (U_{t,K} - \text{Supp}(D_K))(\overline{K})$. As E_t is bounded in $U_t(\overline{K})$ (by construction), the function α_t is bounded on E_t .

Let $c_K = \sum_i n_i [x_{K,i}] \in Z_0^0(X_K)$ whose support does not meet that of D_K . Let $x_{K,i}$ be a closed point in the support of c_K , Z_i its schematic closure in X , $x_{k,i} = X_k \cap Z_i$ and t_i such that $Z_i \subset U_{t_i}$. The same local computation as in the case of a principal divisor shows that

$$(\overline{c_K} \cdot \Delta)_{x_{K,i}} = [K(x_{K,i}) : K] v(g_{t_i}(x_{K,i})) = \sum_{j_i=1}^{s_i} l_i v(g_{t_i}(x_{K,i})).$$

On the other hand, keeping the same notation as in the beginning of the proof,

$$\langle c_K, nD_K \rangle = \sum_i n_i \sum_{j_i=1}^{s_i} l_i \lambda_{nD_K}(x_{\overline{K},j_i}).$$

Consequently,

$$n\delta(c_K, D_K) = \sum_i n_i \sum_{j_i=1}^{s_i} l_i \alpha_{t_i}(x_{\overline{K},j_i}).$$

By construction, the \overline{K} -point $x_{\overline{K},j_i}$ of X_K belongs to E_{t_i} . Denoting by $|\cdot|$ the usual absolute value on \mathbb{R} , and setting

$$B := \max_{t=1, \dots, m} (\sup_{E_t} |\alpha_t|) \in \mathbb{R},$$

we obtain

$$|\delta(c_K, D_K)| \leq \frac{1}{|n|} \sum_i |n_i| [K(x_{K,i}) : K] B = \frac{2B}{|n|} \deg^+ c_K.$$

As the divisor D_K is fixed, the numbers n and B are fixed, and so the right-hand side of the above inequality is bounded if $\deg^+ c_K$ is. \square

Let us note the following property of the pairing $[\ , \]$, and consequently of Néron's pairing.

Proposition 2.6. *Let X_K be a proper geometrically normal and geometrically connected scheme over K . Let c_K be a 0-cycle on X_K which is rationally equivalent to zero. Then*

$$[c_K, D_K] \in \mathbb{Z}$$

for any divisor D_K which is τ -equivalent to zero on X_K and whose support is disjoint from that of c_K .

Proof. As $[\cdot, D_K]$ is \mathbb{Z} -linear, we have to show that if $c_K = (\varphi_K)_* \operatorname{div}_{C_K} f$ for some K -morphism

$$\varphi_K : C_K \rightarrow X_K$$

from a proper normal connected curve C_K to X_K , and some nonzero $f \in K(C_K)$, then

$$[c_K, D_K] \in \mathbb{Z}.$$

As R is excellent, there exists a proper flat *regular* model C/R of C_K . On the other hand, let us consider a proper flat normal semi-factorial model X/R of X_K . After replacing C by a desingularisation of the graph of the rational map $C \dashrightarrow X$ induced by φ_K , we can suppose that φ_K extends to an R -morphism $\varphi : C \rightarrow X$. If Δ is a divisor on X which is τ -equivalent to zero and such that $\Delta_K = nD_K$ for some integer $n \neq 0$, then

$$[c_K, D_K] := \frac{1}{n} (\overline{(\varphi_K)_* \operatorname{div}_{C_K} f} \cdot \Delta) = \frac{1}{n} (\overline{\operatorname{div}_{C_K} f} \cdot \varphi^* \Delta)$$

by the projection formula. Let us write

$$\operatorname{div}_C f = \overline{\operatorname{div}_{C_K} f} - V \quad \text{and} \quad \varphi^* \Delta = \overline{(\varphi_K)^* \Delta_K} - W$$

for some vertical divisors V and W on C/R . Denote by $\Gamma_1, \dots, \Gamma_\nu$ the reduced irreducible components of C_k , by M the intersection matrix associated to C_k (as defined in the introduction), and by $\rho : \operatorname{Pic}(C) \rightarrow \mathbb{Z}^\nu$ the degree homomorphism $(E) \mapsto (E \cdot \Gamma_i)_{i=1, \dots, \nu}$. Following [BLR] 9.2/13, the divisor E on the R -curve C is algebraically equivalent to zero if and only if (E) belongs to the kernel of ρ . Therefore the τ -equivalence relation and the algebraic equivalence relation on C/R are the same, and the linear equivalence classes of $\varphi^* \Delta$ and $\operatorname{div}_C f$ belongs to the kernel of ρ . Thus we get:

$$\rho(\overline{\operatorname{div}_{C_K} f}) = \rho(V) = MV \quad \text{and} \quad \rho(\overline{(\varphi_K)^* \Delta_K}) = \rho(W) = MW,$$

where we have identified a vertical divisor on C/R with an element of \mathbb{Z}^ν . Next, we use that the matrix M is *symmetric* to obtain

$$(\overline{\operatorname{div}_{C_K} f} \cdot W) = {}^t W \rho(\overline{\operatorname{div}_{C_K} f}) = {}^t W M V = {}^t V M W = (\overline{(\varphi_K)^* \Delta_K} \cdot V).$$

Then it follows that

$$[c_K, D_K] = \frac{1}{n} (\overline{(\varphi_K)^* \Delta_K} \cdot \operatorname{div}_C f) = (\overline{(\varphi_K)^* D_K} \cdot \operatorname{div}_C f) \in \mathbb{Z}.$$

□

Remark 2.7. Let us keep the notation of the proof of 2.6. If the curve C_K is *geometrically* normal and *geometrically* connected, the pairing $[\cdot, \cdot]$ is defined on C_K and

$$(\overline{(\varphi_K)^* D_K} \cdot \operatorname{div}_C f) = [(\varphi_K)^* D_K, \operatorname{div}_{C_K} f].$$

In other words, in this case, the proof consists in using the functoriality of the pairing $[\cdot, \cdot]$, then showing that it is *symmetric* for curves, and finally to apply the definition of the pairing for a principal divisor. The symmetry property of Néron's pairing $\langle \cdot, \cdot \rangle$ for such a curve is well known: e.g. see [La] 11.3.6 et 11.3.7. But here, there is no reason for the curve C_K coming from the rational equivalence relation to satisfy the above *geometric* hypothesis. So we could not use directly the properties of the pairing $\langle \cdot, \cdot \rangle$. However, over an excellent discrete valuation ring, there is no need of these geometric hypothesis on C_K for the existence of the regular model C/R . So we have been able to prove the proposition for the pairing $[\cdot, \cdot]$, and thus also for Néron's pairing $\langle \cdot, \cdot \rangle$ thanks to Theorem 2.3.

3 Duality and algebraic equivalence for models of abelian varieties

3.1 Grothendieck's duality for Néron models

Let us recall here Grothendieck's duality theory for Néron models of abelian varieties, as developed in [SGA 7] VII-VIII-IX.

Let R be a discrete valuation ring with perfect residue field k and function field K . Let A_K be an abelian variety over K , with dual A'_K . Let A/R , A'/R be the Néron models of A_K , A'_K , and Φ_A , $\Phi_{A'}$ be the étale k -group schemes of connected components of the special fibers A_k , A'_k .

By definition, the abelian variety A_K represents the identity component $\text{Pic}_{A_K/K}^0$ of the Picard functor, and the canonical isomorphism $A_K = \text{Pic}_{A_K/K}^0$ is given by the Poincaré sheaf \mathcal{P}_K on $A_K \times_K A'_K$ birigidified along the unit sections of A_K and A'_K . Now, this sheaf is canonically endowed with the structure of a *biextension* of (A_K, A'_K) by $\mathbb{G}_{m,K}$ (*loc. cit.* VII 2.9.5). Then the duality theory for Néron models is to understand how this biextension *extends* at the level of Néron models. In this view, Grothendieck attached to \mathcal{P}_K a canonical pairing

$$\langle \cdot, \cdot \rangle : \Phi_A \times_k \Phi_{A'} \rightarrow \mathbb{Q} / \mathbb{Z},$$

which measures the obstruction to extend \mathcal{P}_K as a biextension of (A, A') by $\mathbb{G}_{m,R}$. The duality statement is: *this pairing is a perfect duality* (*loc. cit.* IX 1.3). As mentioned in the introduction, it has been proved in various situations, including the semi-stable case (Grothendieck *loc. cit.* IX 11.4 and Werner [W]) and the mixed characteristic case (Bégueri [Be]). In general, the duality statement remains a conjecture.

Another way to state the duality is the following (e.g. see [B] 4.1). As the component group of the special fiber of the identity component $(A')^0$ of A' is trivial, the Poincaré biextension \mathcal{P}_K extends to the product $A \times_R (A')^0$. Then it defines a canonical morphism from $(A')^0$ to the fppf sheaf of extensions of A by $\mathbb{G}_{m,R}$

$$(A')^0 \rightarrow \underline{\text{Ext}}^1(A, \mathbb{G}_{m,R}).$$

The duality statement becomes: this homomorphism is bijective. Furthermore, denoting by \mathcal{G} the Néron model of $\mathbb{G}_{m,K}$ over R , the sheaf $\underline{\text{Ext}}^1(A, \mathbb{G}_{m,R})$ injects into the sheaf $\underline{\text{Ext}}^1(A, \mathcal{G})$, which is represented by A' . At this point, note that the duality statement holds if and only if it holds after the base change $R \rightarrow \widehat{R^{\text{sh}}}$, where $\widehat{R^{\text{sh}}}$ is the completion of the strict henselization of R . Thereby we can assume R to be complete with algebraically closed residue field. Then, the duality statement is equivalent to the bijectivity of the homomorphism of abstract groups

$$(A')^0(R) \rightarrow \text{Ext}^1(A, \mathbb{G}_{m,R}).$$

In other words, over a complete discrete valuation ring R with algebraically closed residue field, the duality for A and A' can be stated as follows: *the group $(A')^0(R)$ parametrizes the extensions of A by $\mathbb{G}_{m,R}$.*

Now, it is always possible to compactify A into a proper flat normal R -scheme \overline{A} such that the canonical map $\text{Pic}(\overline{A}) \rightarrow \text{Pic}(A)$ is surjective: see [P] 2.16. As \overline{A}/R is proper and flat, it makes sense to speak about algebraic equivalence on \overline{A} using the identity component of the Picard functor $\text{Pic}_{\overline{A}/R}^0$, as defined in subsection 2.1. Our goal in this section is to understand the duality from the viewpoint of algebraic equivalence, starting from the canonical isomorphism $A'_K = \text{Pic}_{A_K/K}^0$. To do this, we need the following definitions.

\mathbb{Q} -divisors and τ -equivalence. Let Z be a *normal* locally noetherian scheme, so that the canonical homomorphism from the group of divisors on Z into the one of 1-codimensional cycles is *injective* ([EGA IV]₄ 21.6.9 (i)). A 1-codimensional cycle C on Z is said to be a \mathbb{Q} -divisor if there exists $n \in \mathbb{Z} \setminus \{0\}$ such that nC is a divisor.

Let $Z \rightarrow T$ be a proper morphism of schemes, with Z locally noetherian and normal. A \mathbb{Q} -divisor C on Z is said to be τ -equivalent to zero if there exists $n \in \mathbb{Z} \setminus \{0\}$ such that nC is a divisor on Z which is τ -equivalent to zero.

Then, assuming the base R to be a complete discrete valuation ring with algebraically closed residue field, we obtain the following formulation of the duality statement for A and A' :

the group $(A')^0(R)$ parametrizes the \mathbb{Q} -divisors on \overline{A} which are τ -equivalent to zero

(Theorem 3.4 below). When \overline{A} is locally factorial, e.g. regular, this amounts to say that there is a canonical isomorphism

$$(A')^0(R) \xrightarrow{\sim} \text{Pic}_{\overline{A}/R}^{\tau}(R).$$

3.2 About nonrational 0-cycles on abelian varieties

To investigate algebraic equivalence on \overline{A} (above notation), we need some preparation about *nonrational* 0-cycles on A_K , especially those which are supported by *inseparable* points over K .

Let K be a field and let A_K be an abelian variety over K . Let d be a positive integer and let $\text{Hilb}_{A_K/K}^d$ be the Hilbert scheme of points of degree d on A_K . The Grothendieck-Deligne norm map

$$\sigma_d : \text{Hilb}_{A_K/K}^d \rightarrow A_K^{(d)}$$

defined in [SGA 4] XVII page 184 (see also [BLR] pages 252-254) maps $\text{Hilb}_{A_K/K}^d$ to the d -fold *symmetric* product $A_K^{(d)}$. On the other hand, the map

$$A_K^d \rightarrow A_K, \quad (x_1, \dots, x_d) \mapsto x_1 + \dots + x_d,$$

induces a map

$$m_d : A_K^{(d)} \rightarrow A_K.$$

Let us set

$$\mathcal{S}_d := m_d \circ \sigma_d : \text{Hilb}_{A_K/K}^d \rightarrow A_K.$$

Let $a_K \in A_K$ be a closed point of degree d , that is to say, the residue field extension $K(a_K)/K$ has degree d . It corresponds to a rational point $h(a_K) \in \text{Hilb}_{A_K/K}^d(K)$. We will need an explicit description of its image $\mathcal{S}_d(h(a_K)) \in A_K(K)$, when considered as an element of $A_{\overline{K}}(\overline{K})$ (\overline{K} is an algebraic closure of K).

Let us consider the artinian \overline{K} -scheme $a_K \otimes_K \overline{K}$. It is supported by some $a_j \in A_{\overline{K}}(\overline{K})$, $j = 1, \dots, s$, where s is the separable degree of $K(a_K)/K$. The length of each local component of $a_K \otimes_K \overline{K}$ is equal to the inseparable degree of $K(a_K)/K$, and will be denoted by l . So the effective 0-cycle associated to $a_K \otimes_K \overline{K}$ is

$$\sum_{j=1}^s l[a_j] \in Z_0(A_{\overline{K}}).$$

We are going to show that

$$\mathcal{S}_d(h(a_K)) = \sum_{j=1}^s l a_j \in A_{\overline{K}}(\overline{K}).$$

Note that when $K(a_K)/K$ is separable, it follows from Galois descent that the right-hand-side of the equality belongs to $A_K(K)$. However, in the inseparable case, we need the K -morphism \mathcal{S}_d and the above claimed equality.

Lemma 3.1. *Let C be an artinian algebra over an algebraically closed field \overline{K} . Let C_1, \dots, C_s be the local components of C , with respective lengths l_1, \dots, l_s , and let $u_j : C_j \rightarrow \overline{K}$ be the canonical surjection from C_j to its residue field. Then, for all $c = (c_1, \dots, c_s) \in C$, the following formula holds for the norm of c over \overline{K} :*

$$N_{C/\overline{K}}(c) = \prod_{i=1}^s (u_j(c_j))^{l_j}.$$

Proof. We can assume that C is local, with length l . Let \mathfrak{m} be the maximal ideal of C . Let n be the smallest integer such that $\mathfrak{m}^n = 0$. Choose a basis $\mathcal{E} = \mathcal{E}_0 \amalg \dots \amalg \mathcal{E}_{n-1}$ of C over \overline{K} which is adapted to the filtration

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \dots \subset \mathfrak{m} \subset C,$$

that is, \mathcal{E}_i is contained in $\mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ and induces a basis of the \overline{K} -vector space $\mathfrak{m}^i / \mathfrak{m}^{i+1}$.

Fix $c \in C$ and let M be the matrix of multiplication-by- c in the basis \mathcal{E} . Write $c = \lambda + \epsilon$ with $\lambda \in \overline{K}$ and $\epsilon \in \mathfrak{m}$. Then M is a lower triangular matrix, whose each diagonal entry is λ . Hence $N_{C/\overline{K}}(c) = \lambda^l$, as required. \square

Let us use the lemma to compute $\sigma_d(h(a_K))$, considered as an element of $A_{\overline{K}}^{(d)}(\overline{K})$. Set $C := \Gamma(a_K \otimes_K \overline{K})$ and $\text{TS}_{\overline{K}}^d(C) := (C^{\otimes d})^{\mathfrak{S}_d} \subseteq C^{\otimes d}$ where \mathfrak{S}_d is the symmetric group acting on $C^{\otimes d}$ by permuting factors. By definition, the point $\sigma_d(h(a_K)) \in (a_K \otimes_K \overline{K})^{(d)}(\overline{K}) \subset A_{\overline{K}}^{(d)}(\overline{K})$ corresponds to the unique \overline{K} -algebra homomorphism

$$\text{TS}_{\overline{K}}^d(C) \rightarrow \overline{K}, \quad c^{\otimes d} \mapsto N_{C/\overline{K}}(c).$$

Now, from the lemma, this homomorphism is induced by the point

$$(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_s, \dots, a_s) \in A_{\overline{K}}^d(\overline{K}),$$

where a_j is repeated l times.

Next, the element $\mathcal{S}_d(h(a_K)) \in A_{\overline{K}}(\overline{K})$ is just the sum

$$m_d(\sigma_d(h(a_K))) = \sum_{j=1}^s l a_j \in A_{\overline{K}}(\overline{K}),$$

as claimed.

Notation 3.2. *The above K -morphisms \mathcal{S}_d induce a homomorphism*

$$\mathcal{S} : Z_0(A_K) \longrightarrow A_K(K)$$

from the group of 0-cycles on A_K to the one of K -rational points: if $a_K \in A_K$ is a closed point of degree d , defining $h(a_K) \in \text{Hilb}_{A_K/K}^d(K)$, then $\mathcal{S}(a_K) := \mathcal{S}_d(h(a_K))$.

We will also need to ‘translate divisors on A_K by nonrational points’.

Let $\text{Div}_{A_K/K}$ be the scheme of relative effective divisors on A_K . Fix a positive integer d and consider the map

$$A_K^d \times_K \text{Div}_{A_K/K} \rightarrow \text{Div}_{A_K/K}$$

which is given by the functorial formula

$$((a_1, \dots, a_d), D) \mapsto D_{a_1} + \dots + D_{a_d},$$

where D_a is obtained from D by translation by the section a . By symmetry, it induces a map

$$A_K^{(d)} \times_K \operatorname{Div}_{A_K/K} \rightarrow \operatorname{Div}_{A_K/K}.$$

By composing with the norm map σ_d , the latter gives rise to a map

$$\operatorname{Hilb}_{A_K/K}^d \times_K \operatorname{Div}_{A_K/K} \rightarrow \operatorname{Div}_{A_K/K}.$$

Let $a_K \in A_K$ be a closed point of degree d and D_K be an effective divisor on A_K . Denote by $(D_K)_{a_K} \in \operatorname{Div}_{A_K/K}(K)$ the image of $(h(a_K), D_K)$ by the previous arrow. As above, write

$$\sum_{r=1}^d [a_{\overline{K}, r}]$$

for the 0-cycle associated to $a_K \otimes_K \overline{K}$. In this expression, repetitions are allowed. Then, using the above computation of $\sigma_d(h(a_K))$, we see that $(D_K)_{a_K}$, as an element of $\operatorname{Div}_{A_{\overline{K}/\overline{K}}}(\overline{K})$, is equal to

$$\sum_{r=1}^d (D_{\overline{K}})_{a_{\overline{K}, r}}.$$

When a_K is étale over K , it is easy to see that the last divisor descends on A_K . But this turns out to be true in general because of the above construction. Moreover, this description shows that the formation of $(D_K)_{a_K}$ is additive in D_K . We can thus associate a divisor $(D_K)_{a_K}$ on A_K to *any* divisor D_K in the following way: identifying divisors on A_K with 1-codimensional cycles, first use the above to define $(D_K)_{a_K}$ when D_K is a prime cycle, and then extend by \mathbb{Z} -linearity.

Notation 3.3. *If c_K is a 0-cycle on A_K and D_K a divisor on A_K , define the divisor $(D_K)_{c_K}$ on A_K by \mathbb{Z} -linearity from the above situation where c_K is a closed point.*

3.3 Algebraic equivalence on semi-factorial compactifications

Using the existence of semi-factorial compactifications ([P] 2.16), the notion of τ -equivalence (subsection 2.1), and of \mathbb{Q} -divisors (subsection 3.1), we are going to prove the following result.

Theorem 3.4. *Let R be a complete discrete valuation ring with algebraically closed residue field k and function field K . Let A_K be an abelian variety over K , with dual A'_K . Let A (resp. A') be the Néron model of A_K (resp. A'_K) over R . Let \overline{A} be a proper flat normal model of A_K over R , equipped with an open immersion $A \rightarrow \overline{A}$, such that the induced map $\operatorname{Pic}(\overline{A}) \rightarrow \operatorname{Pic}(A)$ is surjective. Then the duality statement of [SGA 7] IX 1.3 is equivalent to the following:*

Let $a'_K \in A'_K(K)$ representing the linear equivalence class of a divisor D'_K on A_K . Then a'_K extends to a section of the identity component $(A')^0$ if and only if D'_K can be extended to a \mathbb{Q} -divisor on \overline{A} which is τ -equivalent to zero.

When \overline{A} is locally factorial (e.g. regular), this duality statement reduces to the following:

The canonical map $A'_K(K) \xrightarrow{\sim} \operatorname{Pic}_{A_K/K}^0(K)$ induces a bijection

$$(A')^0(R) \xrightarrow{\sim} \operatorname{Pic}_{A/R}^\tau(R).$$

The starting point is the link between Grothendieck's pairing recalled in subsection 3.1, and Néron's pairing, which has been established by Bosch and Lorenzini: Grothendieck's pairing is the *specialization* of Néron's pairing.

Theorem 3.5. ([BL] 4.4) *Keep the notation of Theorem 3.4. Moreover, let Φ_A (resp. $\Phi_{A'}$) be the group of connected components of A_k (resp. A'_k). On the one hand, consider Grothendieck's pairing [SGA 7] IX 1.3*

$$\langle \cdot, \cdot \rangle : \Phi_A \times \Phi_{A'} \rightarrow \mathbb{Q} / \mathbb{Z},$$

and on the other hand, consider Néron's pairing [N]

$$\langle \cdot, \cdot \rangle : Z_0^0(A_K) \times \text{Div}^0(A_K) \rightarrow \mathbb{Q}$$

(defined for (c_K, D_K) when the supports of c_K and D_K are disjoint).

Let $(a, a') \in \Phi_A \times \Phi_{A'}$. Fix a point $a_K \in A_K(K)$ specializing to a , and a divisor $D'_K \in \text{Div}^0(A_K)$ whose image in $A'_K(K)$ specializes to a' . Assume that a_K and 0_K do not belong to the support of D'_K . Then

$$\langle a, a' \rangle = - \langle [a_K] - [0_K], D'_K \rangle \pmod{\mathbb{Z}}.$$

Until the end of this section 3, we fix a complete discrete valuation ring R with algebraically closed residue field k and fraction field K .

Next proposition is a key result about the pairing $[\cdot, \cdot]$ defined in subsection 2.1.

Proposition 3.6. *Let X_K be a proper geometrically normal and geometrically connected scheme over K . Let X be a proper flat normal semi-factorial model of X_K over R . Let ν be the number of irreducible components of the special fiber X_k . There exists some 0-cycles of degree zero $c_{K,1}, \dots, c_{K,\nu}$ on X_K , with the following property:*

if D_K is a divisor on X_K which is τ -equivalent to zero, whose support is disjoint from those of the $c_{K,i}$, and if $[c_{K,i}, D_K]$ is an integer for all $i = 1, \dots, \nu$, then there exists a \mathbb{Q} -divisor on X which is τ -equivalent to zero, with generic fiber D_K .

Proof. Let U be the open subset of X consisting of the regular points. As X is normal, for any irreducible closed subset C of codimension 1 in X , the intersection $C \cap U$ is a dense open subset of C . Furthermore, for any 1-codimensional cycle C on X , the restriction $C|_U$ is a divisor on U .

Next, let $\Gamma_1, \dots, \Gamma_\nu$ be the reduced irreducible components of X_k . Let ξ_1, \dots, ξ_ν be the generic points of $\Gamma_1, \dots, \Gamma_\nu$. Set $d_i := \text{length}(\mathcal{O}_{X_k, \xi_i})$. From [R] 7.1.2, there exists, for all $i = 1, \dots, \nu$, an R -immersion $u_i : Z_i \rightarrow U$, with Z_i finite and flat over R , with rank d_i , such that $u_{i,k}(Z_{i,k})$ is a point $x_{i,k}$ of Γ_i . Then the intersection multiplicity of Z_i and $\Gamma_j \cap U$ is equal to 1 if $i = j$, and 0 otherwise. In particular, the generic fiber of Z_i is a closed point $x_{K,i} \in U_K$ of degree d_i . Moreover, as Z_i is proper over R , the immersion $Z_i \rightarrow X$ is closed. Finally, setting $d := \text{pgcd}(d_i, i = 1, \dots, \nu)$, an appropriate \mathbb{Z} -linear combination of the $x_{K,i}$ provides a 0-cycle c_K on X_K of degree d . We set

$$c_{K,i} := [x_{K,i}] - \frac{d_i}{d} c_K \in Z_0^0(X_K).$$

Let $D_K \in \text{Div}^\tau(X_K)$ whose support is disjoint from those of the $c_{K,i}$. Let Δ be a divisor on X which is τ -equivalent to zero, and n be a nonzero integer such that $\Delta_K = nD_K$. Denoting by $\overline{D_K}$ the schematic closure of D_K in X , we can view Δ as a 1-codimensional cycle on X , and write

$$\Delta = n\overline{D_K} + \sum_{i=1}^{\nu} n_i \Gamma_i$$

for some integers n_1, \dots, n_ν . Set $V := \sum_{i=1}^\nu n_i \Gamma_i$. As the schematic closures $\overline{c_{K,i}}$ of the $c_{K,i}$ in X are contained in U (by construction), the following computation is valid:

$$\begin{aligned} \overline{c_{K,i}} \cdot \Delta &= n(\overline{c_{K,i}} \cdot \overline{D_K}) + (\overline{x_{K,i}} \cdot V) - \frac{d_i}{d}(\overline{c_K} \cdot V) \\ &= n(\overline{c_{K,i}} \cdot \overline{D_K}) + n_i - \frac{d_i}{d}(\overline{c_K} \cdot V). \end{aligned}$$

Assume that $[c_{K,i}, D_K]$ belongs to \mathbb{Z} . Then, the left-hand side of the above equality belongs to $n\mathbb{Z}$. Consequently, there exists $r_i \in \mathbb{Z}$ such that

$$nr_i = n_i - \frac{d_i}{d}(\overline{c_K} \cdot V).$$

Now, consider the vertical cycle (with integral coefficients)

$$W := (\overline{c_K} \cdot V) \frac{1}{d} [X_k].$$

By definition,

$$V - W = n \sum_{i=1}^\nu r_i \Gamma_i, \quad \text{that is,} \quad \Delta - W = n(\overline{D_K} - \sum_{i=1}^\nu r_i \Gamma_i).$$

The cycle $D := \overline{D_K} - \sum_{i=1}^\nu r_i \Gamma_i$ is equal to D_K on the generic fiber. This is a \mathbb{Q} -divisor on X which is τ -equivalent to zero because dnD is a divisor on X which is τ -equivalent to zero. \square

Keep the notation of Proposition 3.6. Even if X/R admits a section, so that d is equal to 1, the closed point $x_{K,i}$ is *not* rational as soon as the special fiber X_k is not reduced at the generic point of the irreducible component Γ_i . Therefore, if we want to combine Theorem 3.5 and Proposition 3.6 when $X = \overline{A}$ (notation of Theorem 3.4), we need to compare the values of Néron's pairing on the *abelian variety* A_K for 0-cycles which are supported by nonrational points, with its values for 0-cycles of the form $[a_K] - [0_K]$, with $a_K \in A_K(K)$. Here we will use the constructions of subsection 3.2. Moreover, we will need some biduality argument, involving a precise Poincaré divisor P , that we introduce now.

Consider an abelian variety A_K over K , with dual A'_K . Let \mathcal{P} be a Poincaré sheaf on $A_K \times_K A'_K$, birigidified along $0_K \in A_K(K)$ and $0'_K \in A'_K(K)$. Choose a Poincaré divisor P on $A_K \times_K A'_K$, that is, the invertible sheaf $\mathcal{O}_{A_K \times_K A'_K}(P)$ is isomorphic to \mathcal{P} . Replacing P by a linearly equivalent divisor if necessary, one can assume that the restrictions $P|_{0_K \times_K A'_K}$ and $P|_{A_K \times_K 0'_K}$ are well-defined and equal to zero (see Remark 3.8 below). Then, for all $a'_K \in A'_K(K)$, the point $(0_K, a'_K)$ does not belong to the support of P . In particular, the closed subscheme $A_K \times_K a'_K$ of $A_K \times_K A'_K$ is not contained in the support of P , and the divisor $P|_{A_K \times_K a'_K}$ on $A_K \times_K a'_K \simeq A_K$ is well-defined. It will be denoted by $P_{a'_K}$. Similarly, for all $a_K \in A_K(K)$, the divisor $P|_{a_K \times_K A'_K}$ on $a_K \times_K A'_K \simeq A'_K$ is well-defined, and will be denoted by P_{a_K} :

$$P_{a'_K} := P|_{A_K \times_K a'_K}, \quad P_{a_K} := P|_{a_K \times_K A'_K}.$$

Note that the point 0_K is not in the support of $P_{a'_K}$, otherwise $(0_K, a'_K)$ would be in that of P , and a'_K would be in that of $P_{0_K} = 0$. By a symmetric argument, the point $0'_K$ is not in the support of P_{a_K} .

Proposition 3.7. *Let $c_K \in Z_0^0(A_K)$ and $a'_K \in A'_K(K)$. Assume:*

1. the supports of c_K and $P_{a'_K}$ are disjoint;
2. the rational point $\mathcal{S}(c_K)$ (Notation 3.2) does not belong to the support of $P_{a'_K}$.

Then the following relation between values of Néron's pairing on A_K is true:

$$\langle c_K, P_{a'_K} \rangle_{A_K} \equiv \langle [\mathcal{S}(c_K)] - [0_K], P_{a'_K} \rangle \pmod{\mathbb{Z}}.$$

Proof. Write $c_K = c_K^+ - c_K^-$ where c_K^+ and c_K^- are positive 0-cycles with disjoint supports. Let L/K be a finite field extension such that

$$c_K^+ \otimes_K L = \sum_{r=1}^d [a_{r,+}] \quad \text{and} \quad c_K^- \otimes_K L = \sum_{r=1}^d [a_{r,-}]$$

where $d := \deg c_K^+ = \deg c_K^-$ and with $a_{r,+}$, $a_{r,-}$ in $A_L(L)$ (repetitions allowed). Computing Néron's pairing with normalized valuations, we get

$$\langle c_K, P_{a'_K} \rangle_{A_K} = \frac{1}{e_L} \left\langle \sum_{r=1}^d [a_{r,+}] - \sum_{r=1}^d [a_{r,-}], (P_L)_{a'_L} \right\rangle_{A_L},$$

where P_L is the pull-back of P over L , the point $a'_L \in A'_L(L)$ is the image of $a'_K \in A'_K(K)$ by the inclusion $A'_K(K) \subseteq A'_L(L)$, and e_L is the ramification index of L/K . As $(P_L)_{0'_L} = 0$, the *reciprocity law* for Néron's pairing ([La] 11.4.2)¹ asserts that the right-hand side of the equality is equal to

$$\frac{1}{e_L} \langle [a'_L] - [0'_L], \sum_{r=1}^d (P_L)_{a_{r,+}} - \sum_{r=1}^d (P_L)_{a_{r,-}} \rangle_{A'_L}.$$

Now, with Notation 3.3, the divisor $\sum_{r=1}^d (P_L)_{a_{r,+}} - \sum_{r=1}^d (P_L)_{a_{r,-}}$ is precisely the pull-back over L of the divisor P_{c_K} on A'_K . Furthermore, as the map

$$A_L(L) \rightarrow \text{Pic}_{A'_L/L}^0(L)$$

defined by \mathcal{P} is a group homomorphism, the divisors P_{c_K} and $P_{S(c_K)}$ are linearly equivalent on A'_L , and thus on A'_K (because, for example, $\text{Pic}_{A'_K/K}^0(K) \subseteq \text{Pic}_{A'_K/K}^0(L)$). Let $f \in K(A'_K)$ such that $P_{c_K} - P_{S(c_K)} = \text{div}(f)$. As the normalized valuation on K takes values in \mathbb{Z} , the (well-defined) pairing

$$\frac{1}{e_L} \langle [a'_L] - [0'_L], (\text{div}(f))_L \rangle_{A'_L} = \langle [a'_K] - [0'_K], \text{div}(f) \rangle_{A'_K}$$

is an *integer*. Consequently,

$$\langle c_K, P_{a'_K} \rangle_{A_K} \equiv \langle [a'_K] - [0'_K], P_{S(c_K)} \rangle_{A'_K} \pmod{\mathbb{Z}}.$$

As $P_{0_K} = 0$ and $P_{0'_K} = 0$, we conclude by using once again the reciprocity law. \square

Remark 3.8. Fix a point $a'_K \in A'_K(K)$ and cycles $c_{K,1}, \dots, c_{K,\nu} \in Z_0^0(A_K)$. Let us show that there exists a Poincaré divisor P on $A_K \times_K A'_K$ such that $P|_{0_K \times_K A'_K}$ and $P|_{A_K \times_K 0'_K}$ are well-defined and equal to zero, *and* such that the two conditions on supports in statement 3.7 are satisfied for a'_K , P and $c_{K,i}$ for all $i = 1, \dots, \nu$.

Consider the finite set \mathcal{E} whose elements are the following closed points of the product $A_K \times_K A'_K$:

¹Here we use the reciprocity law in the case where the divisorial correspondence is the Poincaré divisor P_L . By using a definition of Néron's pairing relying on the Poincaré biextension (see [Z] §5 or [MT] §2), the reciprocity law for P_L is a direct consequence of the *biduality* of abelian varieties.

1. $(0_K, 0_{K'})$ and $(0_K, a'_K)$;
2. $a_K \times_K a'_K$ if the closed point $a_K \in A_K$ belongs to the support of one of the cycles $c_{K,i}$, or is equal to one of the $\mathcal{S}(c_{K,1}), \dots, \mathcal{S}(c_{K,\nu})$;
3. $a_K \times_K 0'_K$ if a_K is as in 2.

Let \mathcal{P} be a Poincaré sheaf on $A_K \times_K A'_K$, birigidified along $0_K \in A_K(K)$ and $0'_K \in A'_K(K)$. Choose an arbitrary divisor Q such that $\mathcal{O}_{A_K \times_K A'_K}(Q) \simeq \mathcal{P}$. Using a moving lemma on the product $A_K \times_K A'_K$ if necessary ([Li] 9.1.11), one can assume that the support of Q is disjoint from the finite set \mathcal{E} . As $(0_K, 0'_K) \in \mathcal{E}$, the divisors $Q|_{0_K \times_K A'_K}$ and $Q|_{A_K \times_K 0'_K}$ are well-defined, and are principal. Then

$$P := Q - p_2^*(Q|_{0_K \times_K A'_K}) - p_1^*(Q|_{A_K \times_K 0'_K})$$

(where $p_1 : A_K \times_K A'_K \rightarrow A_K$ and $p_2 : A_K \times_K A'_K \rightarrow A'_K$ are the projections) is a Poincaré divisor satisfying $P|_{0_K \times_K A'_K} = 0$ and $P|_{A_K \times_K 0'_K} = 0$.

Now, let a_K be a point of the support of some $c_{K,i}$, or which is equal to some $\mathcal{S}(c_{K,i})$, and assume that a_K belongs to the support $\text{Supp}(P_{a'_K})$ of $P_{a'_K}$ (in other words, one of the conditions of Proposition 3.7 is not satisfied for some $c_{K,i}$). Then $a_K \times_K a'_K \in \text{Supp}(P)$. But $a_K \times_K a'_K \notin \text{Supp}(Q)$ because $a_K \times_K a'_K \in \mathcal{E}$. Next $a_K \times_K a'_K \notin \text{Supp}(p_2^*(Q|_{0_K \times_K A'_K}))$, that is $a'_K \notin \text{Supp}(Q|_{0_K \times_K A'_K})$, or equivalently $(0_K, a'_K) \notin \text{Supp}(Q)$, because $(0_K, a'_K) \in \mathcal{E}$. Hence $a_K \times_K a'_K \in \text{Supp}(p_1^*(Q|_{A_K \times_K 0'_K}))$, $a_K \in \text{Supp}(Q|_{A_K \times_K 0'_K})$ and $a_K \times_K 0'_K \in \text{Supp}(Q)$. The latter is impossible because $a_K \times_K 0'_K \in \mathcal{E}$. In conclusion, the point a_K is not in the support of $P_{a'_K}$, as desired.

We can now interpret Grothendieck's obstruction (subsection 3.1) in terms of relative algebraic equivalence.

Theorem 3.9. *Keep the notation of Theorem 3.4. Moreover, let Φ_A (resp. $\Phi_{A'}$) be group of connected components of A_k (resp. A'_k).*

Let $a' \in \Phi_{A'}$. Lift a' to a point $a'_K \in A'_K(K)$, representing the linear equivalence class of a divisor D'_K on A_K . Then Grothendieck's obstruction

$$\langle \cdot, a' \rangle : \Phi_A \rightarrow \mathbb{Q} / \mathbb{Z}$$

vanishes if and only if D'_K can be extended to a \mathbb{Q} -divisor on \overline{A} which is τ -equivalent to zero.

In particular, when \overline{A} is locally factorial, the obstruction $\langle \cdot, a' \rangle$ vanishes if and only if a'_K can be extended in $\text{Pic}_{\overline{A}/R}^{\tau}(R)$.

Proof. Proposition 3.6 applied with the model \overline{A}/R of A_K provides some 0-cycles of degree zero $c_{K,1}, \dots, c_{K,\nu}$ on A_K . Let P be the Poincaré divisor constructed from the point $a'_K \in A'_K(K)$ and the cycles $c_{K,i}$ in Remark 3.8.

Suppose that the obstruction $\langle \cdot, a' \rangle$ vanishes. As D'_K is linearly equivalent to the well-defined divisor $P_{a'_K}$, it can be extended to a \mathbb{Q} -divisor on \overline{A} which is τ -equivalent to zero if and only if the same is true for $P_{a'_K}$, thereby we can assume that $D'_K = P_{a'_K}$. According to Bosch-Lorenzini's Theorem 3.5, we get

$$\langle [\mathcal{S}(c_{K,i})] - [0_K], P_{a'_K} \rangle \in \mathbb{Z}$$

for all $i = 1, \dots, \nu$. Proposition 3.7 and Theorem 2.3 then imply that

$$[c_{K,i}, P_{a'_K}] \in \mathbb{Z}$$

for all $i = 1, \dots, \nu$. Due to the choice of the $c_{K,i}$, the divisor $P_{a'_K}$ can then be extended to a \mathbb{Q} -divisor on \overline{A} which is τ -equivalent to zero.

Conversely, suppose that there is a \mathbb{Q} -divisor D' on \overline{A} which is τ -equivalent to zero, with generic fiber D'_K . To prove that $\langle \cdot, a' \rangle = 0$, we can assume that 0_K does not belong to the support of D'_K , by adding to D' the divisor of a rational function on \overline{A} if needed. Let n' be a nonzero integer such that $\Delta' := n'D'$ is a divisor on \overline{A} which is τ -equivalent to zero. For each $a_K \in A_K(K)$ which is not in the support of D'_K , we get:

$$[a_K] - [0_K], D'_K = \frac{1}{n'}([\overline{a_K}] - [\overline{0_K}].\Delta') = ([\overline{a_K}] - [\overline{0_K}].D') \in \mathbb{Z}.$$

The first equality holds by definition of the pairing $[\cdot, \cdot]$, and the second one is true because $[\overline{a_K}] - [\overline{0_K}]$ is contained in the regular locus of \overline{A} . Now observe that an element $a \in \Phi_A$ can always be lifted to a point $a_K \in A_K(K)$ which is not in the support of D'_K . Thus, it follows from Theorem 2.3 and Bosch-Lorenzini's Theorem 3.5 that the obstruction $\langle \cdot, a' \rangle$ vanishes.

Assume moreover that \overline{A} is locally factorial, so that any 1-codimensional cycle on \overline{A} is a divisor. In particular, if the obstruction $\langle \cdot, a' \rangle$ vanishes, we have seen that a'_K can be extended in $\text{Pic}_{\overline{A}/R}^\tau(R)$. Conversely, suppose that a'_K can be extended in $\text{Pic}_{\overline{A}/R}^\tau(R)$. As R is strictly henselian, and the sheaf $\text{Pic}_{\overline{A}/R}$ can be defined using the étale topology, the group $\text{Pic}_{\overline{A}/R}(R)$ can be identified with $\text{Pic}(\overline{A})$, which in turn can be identified with the group of divisors on \overline{A} modulo the linear equivalence relation (the scheme \overline{A} being normal). As a consequence, there exists a divisor D' on \overline{A} which is τ -equivalent to zero, and whose generic fiber D'_K is parametrized by a'_K , that is, $(D'_K) = a'_K$. Reasoning as above (with $n' = 1$), we can conclude that $\langle \cdot, a' \rangle = 0$. \square

Proof of Theorem 3.4. By biduality of abelian varieties, Grothendieck's duality statement is equivalent to the following: the obstruction $\langle \cdot, a' \rangle$ vanishes if and only if $a' = 0$. So the theorem in the case where \overline{A} is normal semi-factorial is now clear. Assume moreover that \overline{A} is locally factorial. As the special fiber of \overline{A}/R admits at least one irreducible component with multiplicity 1 (the component containing the unit element of A_k), the restriction morphism

$$\text{Pic}_{\overline{A}/R}^\tau(R) \rightarrow \text{Pic}_{A_K/K}^\tau(K) = A'_K(K)$$

is injective ([R] 6.4.1 3)). On the other hand, the bijection $(A')(R) = A'_K(K)$ induces an inclusion

$$(A')^0(R) \subseteq A'_K(K).$$

The last assertion of the theorem follows. \square

Remark 3.10. From the viewpoint of the theory of the Picard functor $\text{Pic}_{\overline{A}/R}$, Theorem 3.4 is remarkable for two reasons, which come from the properties of the generic fiber $\text{Pic}_{A_K/K}$. To fix the ideas, suppose that \overline{A} is locally factorial, e.g. regular, and cohomologically flat over R . As in [P] section 3, denote by P the schematic closure of A'_K in the algebraic space $\text{Pic}_{\overline{A}/R}$, and by $\tilde{P} \rightarrow P$ its group smoothening. Then the Néron model A' of A'_K is equal to \tilde{P}/F , where F is the schematic closure of the unit section of A'_K in \tilde{P} (*loc. cit.* 3.3). We claim that, when Grothendieck's duality statement is true, the following equalities hold:

$$(\tilde{P})^0(R) = P^0(R) = P^\tau(R).$$

Indeed, the group smoothening $\tilde{P} \rightarrow P$ always induces a bijection on R -points, hence an injection $(\tilde{P})^0(R) \subseteq P^0(R)$. Next, by definition, $P^0(R) \subseteq P^\tau(R)$. Now, let $a' \in P^\tau(R)$. The generic fiber a'_K of a' is an element of $A'_K(K)$, which extends

to a section $b' \in A'(R)$. When Grothendieck's duality holds, Theorem 3.4 shows that $b' \in (A')^0(R)$. Then b' can be lifted to a point $c' \in (\tilde{P})^0(R)$ (*loc. cit.* 3.6). As $c'_K = b'_K = a'_K$ and $a' - c' \in P^\tau(R)$, we get $a' - c' = 0$ ([R] 6.4.1 3), whence $a' \in (\tilde{P})^0(R)$.

The first equality $(\tilde{P})^0(R) = P^0(R)$ says that the group smoothening $\tilde{P} \rightarrow P$ induces an *injection* $\Phi_{\tilde{P}} \rightarrow \Phi_P$ between the groups of connected components of the special fibers, a fact which is not true for a general group smoothening $\tilde{G} \rightarrow G$.

The second equality $P^0(R) = P^\tau(R)$ says that the group $P(R)/P^0(R)$ has no torsion. When P is smooth over R , e.g. the characteristic of k is zero, this precisely means that the Néron-Séveri group of the special fiber \overline{A}_k has *no torsion*. Note that it is well-known that the Néron-Séveri group of a curve or an abelian variety is free, but in general the scheme \overline{A}_k is not of this type.

4 Some calculation for Jacobians

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K . Let X_K be a proper smooth geometrically connected curve over K , and let $J_K := \text{Pic}_{X_K/K}^0$ be its Jacobian. Denote by J (resp. J') the Néron model of J_K (resp. J'_K) over R , and Φ_J (resp. $\Phi_{J'}$) the group of connected components of the special fiber of J/S (resp. J'/S). Theorems 3.5 and 2.3 describe Grothendieck's pairing associated to J_K in terms of intersection multiplicities on some compactification \overline{J} of J . It is natural to wonder if these computations can be replaced by intersection computations *on a proper flat regular model X of X_K* .

Assume that $X_K(K)$ is nonempty. In this case, the curve X_K can be *embedded into J_K* , and can be used to define a classical theta divisor on J_K . Then, using Theorem 3.5, Bosch and Lorenzini described Grothendieck's pairing associated to J_K in terms of the Néron pairing *on X_K* , and so in terms of intersection multiplicities on X , thanks to Gross's and Hriljac's Theorems [G] and [H]. Their precise result is as follows. Let M be the intersection matrix of the special fiber of X/R : if $\Gamma_1, \dots, \Gamma_\nu$ are the irreducible components of X_k equipped with their reduced scheme structure, the $(i, j)^{\text{th}}$ entry of M is the intersection number $(\Gamma_i \cdot \Gamma_j)$. Denote by Φ_M the torsion part of the cokernel of $M : \mathbb{Z}^\nu \rightarrow \mathbb{Z}^\nu$. According to Raynaud's work on the sheaf $\text{Pic}_{X/S}$, there is a canonical isomorphism $\Phi_J = \Phi_M$ (see [BLR] 9.6/1). Now, on the product $\Phi_M \times \Phi_M$, there is the canonical pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle_M : \Phi_M \times \Phi_M &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (\overline{T}, \overline{T'}) &\mapsto ({}^t S/n)M(S'/n') \pmod{\mathbb{Z}} \end{aligned}$$

for any $n, n' \in \mathbb{Z} \setminus \{0\}$ and $S, S' \in \mathbb{Z}^\nu$ such that $MS = nT$, $MS' = n'T'$. Now let $(a, a') \in \Phi_J \times \Phi_{J'}$. By identifying J_K and J'_K with the help of the opposite of the canonical principal polarization defined by a theta divisor, Grothendieck's pairing of a and a' can be computed by the formula

$$\langle a, a' \rangle = \langle a, a' \rangle_M$$

([BL] Theorem 4.6).

Now assume that $X_K(K)$ is empty. Extending K , it is possible to consider a theta divisor on J_K as above, and it is classical that the associated canonical principal polarization descends over K . Using its opposite, one can still identify Φ_J with $\Phi_{J'}$, and thus $\Phi_{J'}$ with Φ_M (as k is algebraically closed, the identification $\Phi_J = \Phi_M$ holds without assuming that $X_K(K)$ is nonempty). Then the authors of [BL] ask if both pairings $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_M$ still coincide in this situation (*loc. cit.* Remark 4.9). In [Lor] Theorem 3.4, Lorenzini gives a positive answer to this

question when the special fiber of X/R admits two irreducible components C_i and C_j with multiplicities d_i and d_j such that $(C_i \cdot C_j) > 0$ and $\gcd(d_i, d_j) = 1$. Here we show that this result still holds if we only assume that *the global gcd of the multiplicities of the irreducible components of X_k is equal to 1*. Note that, due to the hypothesis on R and on X , this global gcd coincide with the *index* of the curve X_K , that is, the smallest positive degree of a divisor on X_K ([R] 7.1.6 1)).

Proposition 4.1. *Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K . Let X_K be a proper smooth geometrically connected curve over K , with index d . Let J_K be the Jacobian of X_K , identified with its dual using the opposite of its canonical principal polarization. Let X/R be a proper flat regular model of X_K . The following relation between Grothendieck's pairing for J_K and the above pairing defined by the intersection matrix M of X_k is true:*

$$d \langle a, a' \rangle = d \langle a, a' \rangle_M.$$

As \langle, \rangle_M is a perfect duality ([BL] Theorem 1.3), we get the following partial answer to Grothendieck's conjecture [SGA 7] IX 1.3 in this case:

Corollary 4.2. *Keep the notation of Proposition 4.1. The kernel of Grothendieck's pairing for J_K is killed by d .*

Remark 4.3. Using [BB] Theorem 2.1 together with the argument of the proof of *loc.cit.* Theorem 4.1, and then applying Theorem 4.7 of [BL], it is possible to see that the kernel of Grothendieck's pairing for J_K is killed by d^2 .

Here are two lemmas to prepare the proof of Proposition 4.1.

Recall that, as R is complete with algebraically closed residue field, a classical result of Lang asserts that the Brauer group of K is zero, whence $\text{Pic}^0(X_K) = J_K(K)$.

Lemma 4.4. *Let $a, a' \in \Phi_J = \Phi_M$, and choose divisors D_K, D'_K on X_K with disjoint supports, such that $a_K := (D_K), a'_K := (D'_K) \in J_K(K) = \text{Pic}^0(X_K)$ specialize to a, a' . The relationship between the pairing \langle, \rangle_M and Néron's pairing on X_K is given by:*

$$\langle a, a' \rangle_M = - \langle D_K, D'_K \rangle \pmod{\mathbb{Z}}.$$

Proof. This is an immediate consequence of the definitions, and of the description of Néron's pairing for the curve X_K in terms of intersection multiplicities on X . Indeed, let $\rho : \text{Pic}(X) \rightarrow \mathbb{Z}^\nu$ be the degree morphism $(Z) \mapsto (Z \cdot \Gamma_i)_{i=1, \dots, \nu}$. Denote by $\overline{D_K}$ the schematic closure of D_K in X . By definition of Raynaud's isomorphism $\Phi_J = \Phi_M$, the image of $\rho(\overline{D_K}) \in \mathbb{Z}^\nu$ in $\mathbb{Z}^\nu / \text{Im } M$ is contained in the torsion part Φ_M , and the resulting element is precisely the image of $a \in \Phi_J$ under the isomorphism. In particular, there are $n, n' \in \mathbb{Z} \setminus \{0\}$ and $S, S' \in \mathbb{Z}^\nu$ such that $MS = n\rho(\overline{D_K}), MS' = n'\rho(\overline{D'_K})$, and by definition of the symmetric pairing \langle, \rangle_M , we get

$$\langle a, a' \rangle_M = ({}^t S' / n') \rho(\overline{D_K}) \pmod{\mathbb{Z}}.$$

Under the identification $\oplus_{i=1}^\nu \mathbb{Z} \Gamma_i \simeq \mathbb{Z}^\nu$, the right-hand side can also be written as an intersection multiplicity:

$$\langle a, a' \rangle_M = \frac{1}{n'} (\overline{D_K} \cdot S') = -\frac{1}{n'} (\overline{D_K} \cdot (n' \overline{D'_K} - S')) \in \mathbb{Q} / \mathbb{Z}.$$

Now, the equality $MS' = n'\rho(\overline{D'_K})$ means that the divisor $n' \overline{D'_K} - S'$ is algebraically equivalent to zero on X/R ([BLR] 9.2/13). Applying Theorem 2.3 to the curve X_K , we conclude that

$$\langle a, a' \rangle_M = -[D_K, D'_K] = -\langle D_K, D'_K \rangle \in \mathbb{Q} / \mathbb{Z}.$$

□

Next, the index d of X_K divides $g - 1$ where g is the genus of X_K ([R] 9.5.1). Let us fix a divisor E of degree d on X_K , and consider the linear equivalence class of divisors of degree $g - 1$ given by $t_K := (g - 1)d^{-1}(E) \in \text{Pic}_{X_K/K}^{g-1}(K)$. The canonical image of the $(g - 1)$ -fold symmetric product $X_K^{(g-1)}$ in $\text{Pic}_{X_K/K}^{g-1}$ can be translated by t_K to a divisor on J_K , that we will denote by Θ . Then, by extending K and reducing to the case where $X_K(K)$ is nonempty, one sees that the canonical principal polarization of J_K can be written explicitly here as $\varphi(a) = -(\Theta_a - \Theta)$, where Θ_a is obtained from Θ by translation by the section a . On the other hand, the formula $x \mapsto (d[x] - E)$ defines a K -morphism $h : X_K \rightarrow J_K$.

Lemma 4.5. *The following diagram of K -morphisms is commutative:*

$$\begin{array}{ccc} & J'_K & \\ -\varphi \nearrow & & \searrow h^* \\ J_K & \xrightarrow{d} & J_K. \end{array}$$

The commutativity can be stated as follows. Let $z \in J_K(\overline{K})$. Let Z be any divisor of degree 0 on $X_{\overline{K}}$, whose linear equivalence class (Z) corresponds to z via the canonical isomorphism $\text{Pic}^0(X_{\overline{K}}) = J_K(\overline{K})$. Then the following relation holds:

$$h^*(\Theta_z - \Theta) = d(Z) \in \text{Pic}^0(X_{\overline{K}}) = J_K(\overline{K}).$$

In particular, there is a nonempty open subset U_K of J_K such that $h^*\Theta_z$ is a well-defined divisor on X_K for all $z \in U_K(K)$, and whose degree does not depend on the point z .

Proof. To check that the diagram is commutative, one can replace K by its algebraic closure, and so we can assume that K is algebraically closed. As the pull-back by the multiplication-by- d on J_K acts as multiplication-by- d on the group $\text{Pic}^0(J_K)$, the lemma then follows from the classical situation where X_K can be embedded into J_K using a rational point of X_K . □

Proof of Proposition 4.1. Let $(a, a') \in \Phi_J \times \Phi_J$. Choose a point $a_K \in J_K(K)$ which specializes to $a \in \Phi_J$. The point a_K corresponds, under the equality $J_K(K) = \text{Pic}^0(X_K)$, to the linear equivalence class of a divisor $D(a)_K$ of degree 0 on X_K . Write $D(a)_K = D(a)_K^+ - D(a)_K^-$ with $D(a)_K^+$ and $D(a)_K^-$ positive with disjoint supports. Let L/K be a finite field extension such that

$$D(a)_K^+ \otimes_K L = \sum_{r=1}^{\alpha} [a_{r,+}] \quad \text{and} \quad D(a)_K^- \otimes_K L = \sum_{r=1}^{\alpha} [a_{r,-}]$$

where $\alpha := \deg D(a)_K^+ = \deg D(a)_K^-$ and with $a_{r,+}$, $a_{r,-}$ in $X_L(L)$ (repetitions allowed).

Next, still denoting by U_K the open subset of J_K provided by Lemma 4.5, one can find some $a'_K, z_K \in U_K(K)$ specializing to $a', 0 \in \Phi_J$, and such that

$$\begin{aligned} da_K, 0_K &\notin \text{Supp}(\Theta_{a'_K} - \Theta_{z_K}) \subseteq J_K \\ \bar{a}_{r,+}, \bar{a}_{r,-} &\notin \text{Supp}((\Theta_{a'_K} - \Theta_{z_K})_L) \subseteq J_L \quad \forall r = 1, \dots, \alpha, \end{aligned}$$

where $\bar{a}_{r,+} := h(a_{r,+})$ and $\bar{a}_{r,-} := h(a_{r,-})$. The points a'_K and z_K correspond to the classes of some divisors $D(a')_K$ and $D(0)_K$ on X_K , under the identification $J_K(K) = \text{Pic}^0(X_K)$. From Lemma 4.5, we get:

$$h^*(\Theta_{a'_K} - \Theta_{z_K}) = d(D(a')_K - D(0)_K) = d(a'_K - z_K)$$

in $\text{Pic}^0(X_K) = J_K(K)$. And by construction, the K -point $d(a'_K - z_K)$ of J_K specializes to $da' \in \Phi_J$. As a consequence, Lemma 4.4 provides the formula:

$$\langle a, da' \rangle_M = -\langle D(a)_K, h^*(\Theta_{a'_K} - \Theta_{z_K}) \rangle_{X_K} \pmod{\mathbb{Z}}$$

(note that $h^*(\Theta_{a'_K} - \Theta_{z_K})$ is a well-defined divisor, and not only a class, because $a'_K, z_K \in U_K(K)$).

Still working with normalized valuations to compute Néron's pairing, and using functoriality, we obtain:

$$\langle a, da' \rangle_M = -\frac{1}{e_L} \left\langle \sum_{r=1}^{\alpha} [\bar{a}_{r,+}] - [\bar{a}_{r,-}], (\Theta_{a'_K} - \Theta_{z_K})_L \right\rangle_{J_L} \pmod{\mathbb{Z}}.$$

where e_L is the ramification index of L/K . Then we apply the reciprocity law for Néron's pairing with the divisorial correspondence $(\delta^*\Theta - p_1^*\Theta - p_2^*\Theta)_L$, where δ , p_1 and $p_2 : J_K \times_K J_K \rightarrow J_K$ are the difference map and the two projections, to get:

$$\langle a, da' \rangle_M = -\frac{1}{e_L} \left\langle [a'_L] - [z_L], \sum_{r=1}^{\alpha} (\Theta_L)_{\bar{a}_{r,+}}^- - (\Theta_L)_{\bar{a}_{r,-}}^- \right\rangle_{J_L} \pmod{\mathbb{Z}}.$$

Here $(\Theta_L)^-$ stands for $[-1]^*(\Theta_L)$.

Now, with Notation 3.3, the divisor $\sum_{r=1}^{\alpha} (\Theta_L)_{\bar{a}_{r,+}}^- - (\Theta_L)_{\bar{a}_{r,-}}^-$ is the pull-back on J_L of the divisor $(\Theta^-)_{h_*D(a)}$ defined on J_K . On the other hand,

$$\begin{aligned} \sum_{r=1}^{\alpha} \bar{a}_{r,+} - \bar{a}_{r,-} &= \sum_{r=1}^{\alpha} (d[a_{r,+}] - E_L) - (d[a_{r,-}] - E_L) \\ &= d(D(a)_L) \in J_K(L) \\ &= da_K \in J_K(K). \end{aligned}$$

Therefore the theorem of the square on J_L shows that the two divisors $(\Theta^-)_{h_*D(a)}$ and $\Theta_{da_K}^- - \Theta^-$ on J_K are linearly equivalent over L , hence also over K ($J'_K(K)$ injects into $J'_L(L)$). From this observation, and the fact that the normalized valuation on K takes values in \mathbb{Z} , we deduce that

$$\langle a, da' \rangle_M = -\langle [a'_K] - [z_K], \Theta_{da_K}^- - \Theta^- \rangle_{J_K} \pmod{\mathbb{Z}}.$$

Applying once more the reciprocity law, we find

$$\langle a, da' \rangle_M = -\langle [da_K] - [0_K], \Theta_{a'_K} - \Theta_{z_K} \rangle \pmod{\mathbb{Z}}.$$

Finally, note that $(\Theta_{a'_K} - \Theta_{z_K}) = -\varphi(a'_K - z_K) \in J'(K)$ and $a'_K - z_K$ specializes to $a' \in \Phi_J$. Consequently, if we use $-\varphi$ to identify J_K with its dual, Theorem 3.5 tells us that

$$-\langle [da_K] - [0_K], \Theta_{a'_K} - \Theta_{z_K} \rangle = \langle da, a' \rangle \pmod{\mathbb{Z}}.$$

Whence

$$\langle a, da' \rangle_M = \langle da, a' \rangle,$$

as claimed. \square

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